Moment Redistribution: Principles and Practice Using ACI 318-02

By Kenneth B. Bondy[1]

ABSTRACT: *This paper describes the fundamentals of inelastic moment redistribution in indeterminate prestressed concrete members, with a particular focus on the new "unified design provisions" of the 2002 ACI Building Code (ACI 318-02) and how they affect moment redistribution. The fundamentals and statics of "secondary moments" in indeterminate prestressed members is discussed and explained in detail. Examples are presented for the analysis and the design of multi-span prestressed concrete beams in accordance with the new code. Recommendations are made for changes to ACI 318 that will modify certain aspects of moment redistribution and secondary moments not adequately addressed in the current edition.*

KEYWORDS: Inelastic, moment redistribution, indeterminate, secondary moments, prestressed concrete, plastic hinge, ACI 318-02

1.0 INTRODUCTION

Moment redistribution provides the designer of continuous prestressed and non-prestressed beams and slabs a valuable tool for cost-efficient design. Understanding and taking advantage of the effects of inelastic behavior in indeterminate members generally permits the designer to reduce *both* the maximum elastic positive *and* negative moments when live load is "skipped" (arranged in patterns that produce maximum possible positive and negative moments at all sections), thus narrowing the envelope of demand moments across the spans, and reducing the amount of reinforcing required for any given factor of safety. Moment redistribution also often permits the "shifting" of moments from cross-sections that are less efficient in resisting moment to those that are more efficient, resulting in further savings in reinforcing. Significant changes have been made to *"Building Code Requirements for Structural Concrete (ACI 318-02) and Commentary (ACI 318R-02)"* that simplify and unify moment redistribution in both prestressed and non-prestressed continuous beams and slabs. This paper will address the fundamentals of moment redistribution and how the new code affects design practices. It will also address the related subject of secondary moments in continuous prestressed concrete members, and how they interact in the moment redistribution process.

2.0 WHAT IS MOMENT REDISTRIBUTION?

Moment redistribution is a term that describes the behavior of an indeterminate concrete member after first yielding occurs at some cross-section of the member. As applied load is increased on an indeterminate member, the response is initially elastic (deflections, moments, shears are linearly proportional to applied load and can be calculated by elastic indeterminate theory) up to the load where yielding first occurs in any cross-section. The applied load producing first yielding at any cross-section is called w_1. Incremental applied load w_2 greater than w_1 is assumed to produce inelastic rotation at the yielded section, but no change in applied moment. Since w_2 produces no incremental moment at the yielded cross-section, incremental moments resisting w_2 are developed at sections *other* than the initially yielded section. Thus after first yielding, moments are *redistributed* to other cross-sections of the member which are still elastic. As w_2 increases, eventually other sections will yield and develop hinges. When enough hinges have developed in any span of the member to make it unstable (a mechanism rather than a flexural member), the member is considered to have failed. The load at which a mechanism forms in any span is called the "limit" load in that span.

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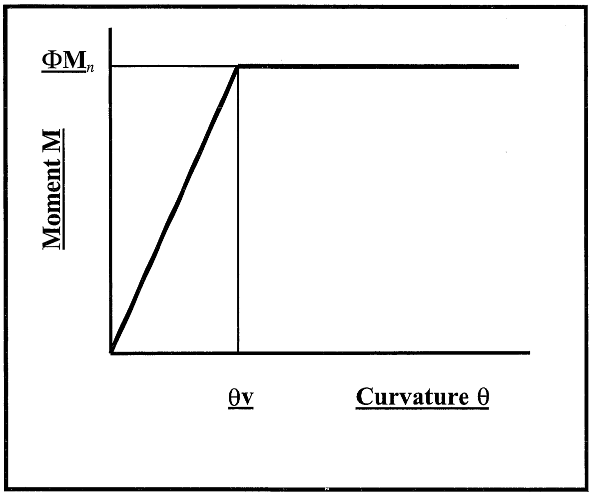


Fig. 1 Idealized bi-linear moment - curvature relationship

Inelastic behavior in indeterminate concrete members is idealized by a bilinear moment-curvature relationship shown in Fig. 1. The cross-section is assumed to respond elastically up to an applied moment of $ØM_n$ at which point the section de-velops a "plastic hinge", and incremental curvature (rotation) occurs at the section with no change in moment.

The inelastic behavior of a continuous member depends on whether yielding first occurs in a negative moment region (at a support) or in the field of the member in the positive moment region. ACI 318 code requirements permit first yielding in either positive or negative moment regions, although this is not immediately obvious.

Moment redistribution is used in the *design* of continuous con-crete members by providing a flexural capacity $ØM_n$ at the negative or positive moment regions of the member (or both) that is less than the moment at the same point calculated by elastic theory. The reduced moment capacities must be statically consistent with moments at other sections of the member under the same loading condition.

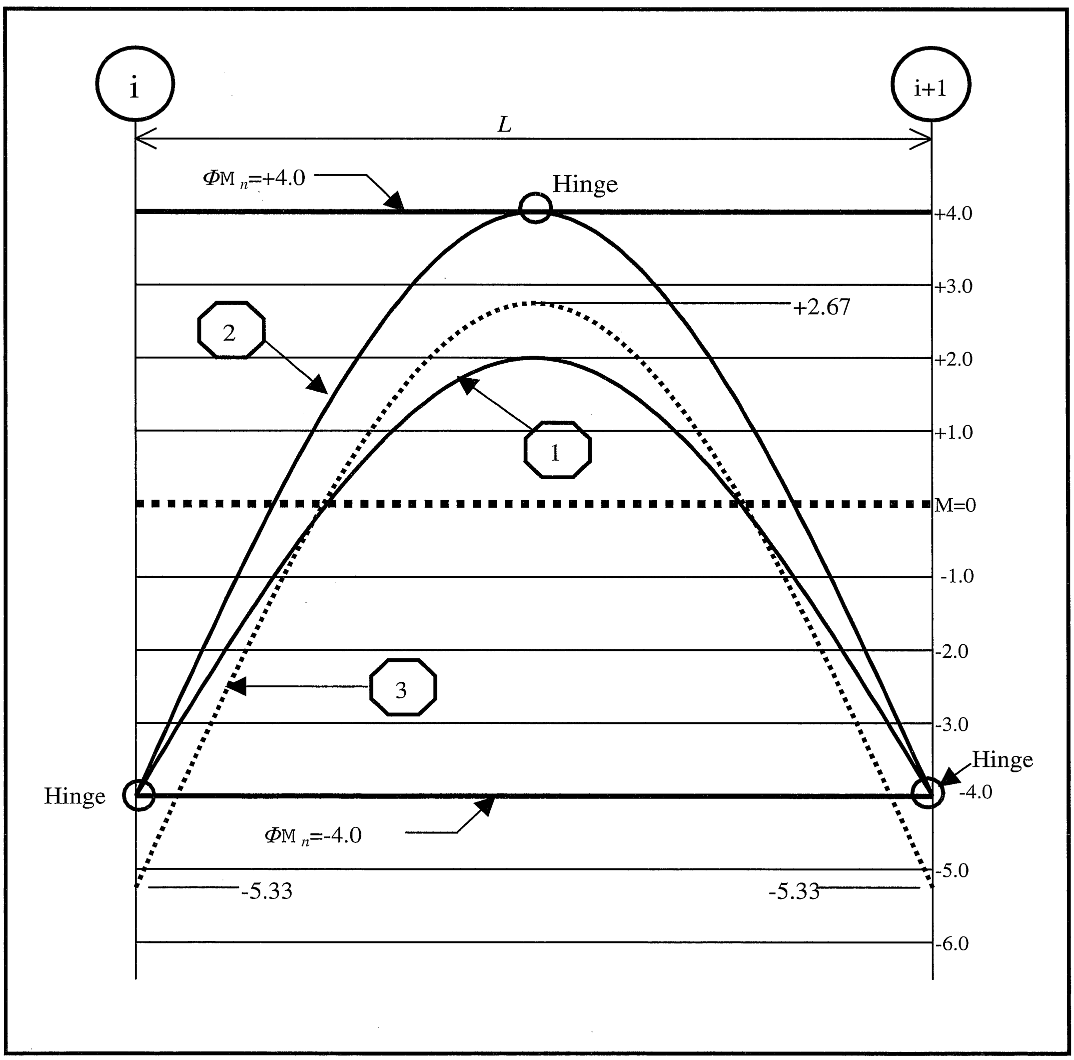


Fig. 2 - Moment redistribution - hinging at supports

3.0 FIRST YIELDING IN NEGATIVE MOMENT REGIONS (SUPPORTS)

Fig. 2 shows a moment diagram for an interior span of length L in a multispan continuous beam with an applied uniform load w per length of beam. The beam has constant positive and negative flexural capacities $ØM_n$ of 4.0 "units". In the elastic range the negative moment at each end of the beam is $-wL^2/12$ and the positive moment at midspan is $+wL^2/24$. Curve 1 is an elastic moment diagram in which the negative moment has just reached the yield moment of -4.0 units. Under this load w_1, the positive moment is +2.0 units. The load which produces this moment diagram (Curve 1) is:

$$w_1 = \frac{12x4}{L^2} = \frac{48}{L^2} \quad \text{.........(1)}$$

As the load increases beyond w_1 the ends of the beams act as hinges undergoing inelastic rotation with no change in moment (the moment stays constant at -4.0 units). The beam is stable and can carry additional load w_2 as a simple-span determinate beam with pinned ends and a constant end moment. This post-yielding rotation at the beam ends is an inelastic behavior, and the amount of rotation possible before failure at the section (crushing of the concrete or tensile rupture of the steel) is a measure of the ductility of the section. At some load w_1+w_2 the midspan yield moment of +4.0 units will be reached and the inelastic moment diagram of Curve 2 in Fig. 2 will be produced. At this point the span has developed three "hinges" and becomes unstable, incapable of resisting addi-tional load in flexure. The limit load $w_{limit}=w_1+ w_2$ which produces the moment diagram of Curve 2 is determined as follows:

$$\frac{w_{\lim it} L^2}{8} = \phi M_n^+ + \phi M_n^- = 8$$

$$w_{\lim it} = \frac{64}{L^2} \quad \text{.........(2)}$$

It is of interest to calculate the moment diagram which would be produced if the beam responded elastically to the limit load w_{limit}. The negative moment at each end of the span is:

$$M_e^- = \left(\frac{64}{L^2}\right)\frac{L^2}{12} = \frac{64}{12} = 5.33 \quad \text{.........(3)}$$

The positive elastic moment is:

$$M_e^+ = \left(\frac{64}{L^2}\right)\frac{L^2}{24} = 2.67 \quad \text{.........(4)}$$

In Fig. 2 Curve 3 is the elastic moment diagram which would be produced by the limit load w_{limit}. At all points where the elastic Curve 3 exceeds the yield capacity $ØM_n$, inelastic behavior is required of the section. The ratio between the elastic moment and the yield moment at any point is a measure of the amount of inelastic behavior required at that point to develop the limit load w_{limit}. The amount of inelastic rotation or "redistribution" required at a section can be defined as:

$$\%R = 100\left(1 - \frac{\phi M_n}{M_e}\right) \quad \text{.........(5)}$$

where M_e is the elastic moment at the section under consideration. In the example of Fig. 2 the amount of redistribution required at the ends of the span to develop the limit load is:

$$\%R = 100\left(1 - \frac{4}{5.33}\right) = 24.9\% \quad \text{.........(6)}$$

4.0 FIRST YIELDING IN POSITIVE MOMENT REGIONS (MIDSPANS)

It is also possible for yielding to occur first at midspan in positive moment regions. Fig. 3 shows such a case, where in this span the flexural capacities at supports and midspan are -3.0 units and +1.0 unit respectively. A load of w_1 produces Curve 1, the elastic moment diagram in which the positive moment has just reached $ØM_n$ =+1.0 at midspan. The magnitude of w_1 is:

$$w_1 = \frac{12x2}{L^2} = \frac{24}{L^2} \quad \text{.........(7)}$$

A plastic hinge develops at midspan under this load, and additional load w_2 beyond w_1 is resisted by two cantilevers off of the left and right supports. The support moments increase until, at a load of $w_{limit}= w_1+w_2$ the support sections yield. The moment diagram produced by the limit load is shown in Fig. 3 as Curve 2. The limit load is:

$$\frac{w_{\lim it} L^2}{8} = \phi M_n^+ + \phi M_n^- = 4$$

$$w_{\lim it} = \frac{32}{L^2} \quad \text{.........(8)}$$

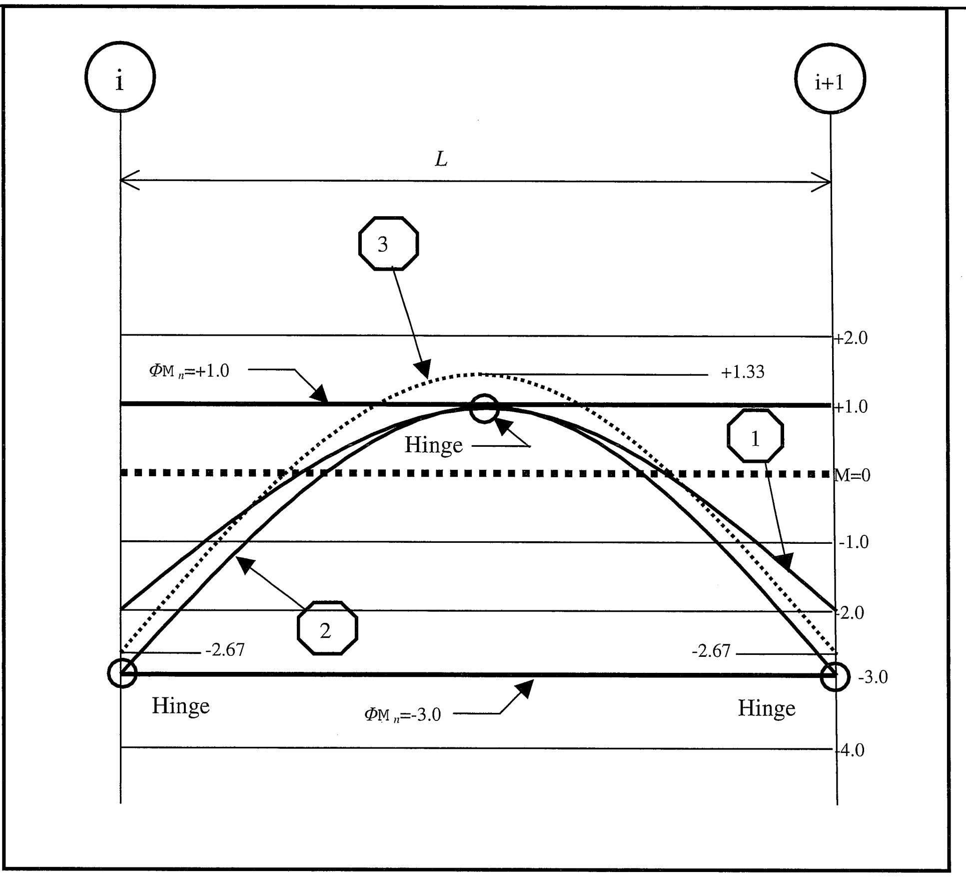


Fig. 3 -Moment redistribution -- hinging at midspan

The elastic negative moment which would be produced by w_{limit} is:

$$M_e^- = \left(\frac{32}{L^2}\right)\frac{L^2}{12} = 2.67 \quad (9)$$

The positive elastic moment that would be produced by w_{limit} is:

$$M_e^+ = \left(\frac{32}{L^2}\right)\frac{L^2}{24} = 1.33 \quad (10)$$

The elastic moment diagram which would be produced by w_{limit} is shown in Fig. 3 as Curve 3. The amount of redistribution required at midspan is:

$$\%R = 100\left(1 - \frac{1}{1.33}\right) = 24.8\% \quad (11)$$

Note in this case there is ***no*** inelastic behavior required at the supports in the negative moment region. All of the inelastic redistribution occurs in the positive moment region at midspan. The elastic negative moment under w_{limit} has been *increased* in magnitude by 12.4% (from -2.67 units to -3.0 units) in order to achieve the reduction in positive moment. Note that the percentage change in the negative support moment (where no inelastic behavior is involved) is significantly smaller (half) than the percent change in positive moment, where inelastic behavior is required.

5.0 EXAMPLES

To demonstrate the mechanics of moment redistribution, a two-span beam model on pinned supports will be used, since it is the simplest indeterminate member available that illustrates all of the necessary aspects of moment redistribution. In order to demonstrate the effects of "secondary moments" through the entire range of loading, the beam model will be assumed to contain both prestressed and non-prestressed reinforcement. Additional factors such as more spans, support width, support stiffness, etc., albeit realistic, add only mathematical complexity to this basic model without illustrating anything fundamental about moment redistribution, therefore these factors

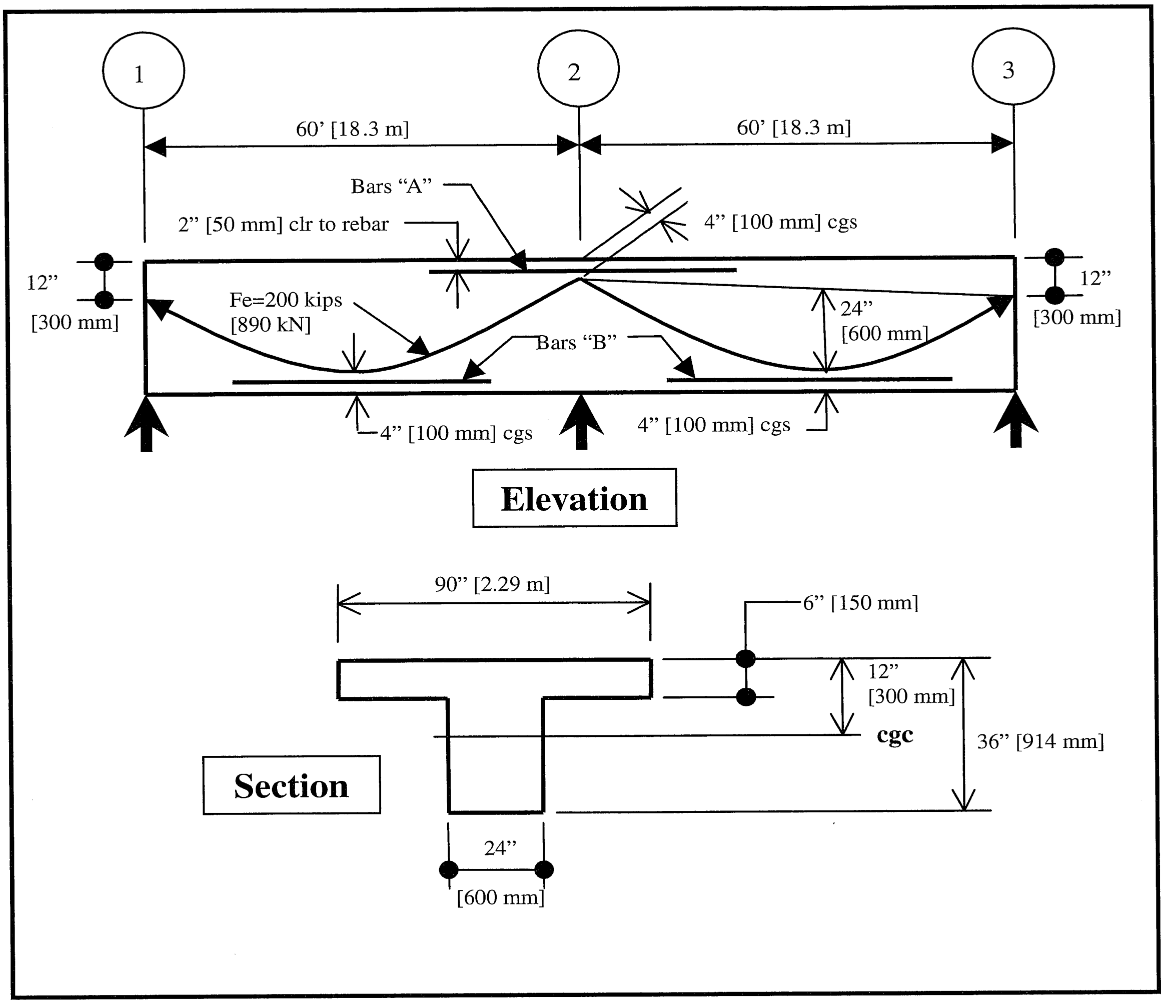


Fig. 4 - Example beam

will be ignored. It is assumed that the reader understands such complexities and can incorporate them as necessary. In order to demonstrate the application of moment redistribution two models with given reinforcement will first be *analyzed*, and then a similar two-span beam will be *designed* in accordance with the new provisions of ACI 318-02.

Assume a two-span beam with geometry and reinforcing as shown in Fig. 4. Tendons are unbonded with parabolic profiles, effective tendon stress f_{se}=173 ksi [1193 MPa], unstressed reinforcing steel is Grade 60 (f_y=414 MPa), concrete f'_c=4,000 psi [27.6 MPa].

5.1 Analysis Case 1:
Bars "A"=2-#6, Bars "B"=2-#8

This example illustrates a beam where yielding develops first in the negative moment region at an interior support. For the given reinforcing, the flexural capacity $\emptyset M_n$ at support 2 is 646 ft-kips [876 kN-m] and at each midspan is 868 ft-kips [1177 kN-m]. In calculating these capacities the tendon stress at nominal strength f_{ps} is 203 ksi [1400 MPa] at support 2 and 233 ksi [1607 MPa] at midspans (ACI 318-02 Eq. 18-4), and $\emptyset$ is 0.9 (both sections are tension-controlled with $\varepsilon_t > 0.005$, see 10.3).

When a statically *determinate* beam is prestressed, the moment at any cross-section produced by the prestress force is simply the prestress force multiplied by the distance ("eccentricity") between the point of application of the force and the geometric centroid of the cross-section. This moment is often called the "primary moment". External reactions in determinate beams are unaffected by prestressing since they can be determined solely by the statics of externally applied loads (thus the term "determinate"). When a statically *indeterminate* beam is prestressed, however, the internal prestressing force can cause changes in external reactions, producing addi-tional moments at beam cross-sections. These additional mo-ments are often called "secondary moments".

Consider the beam shown in Fig. 4. Assume that the beam is weightless (no applied external dead or live loads). If the center support at grid 2 was removed, the beam would span between the external supports at grids 1 and 3. Since the primary moments are predominantly negative (producing flexural

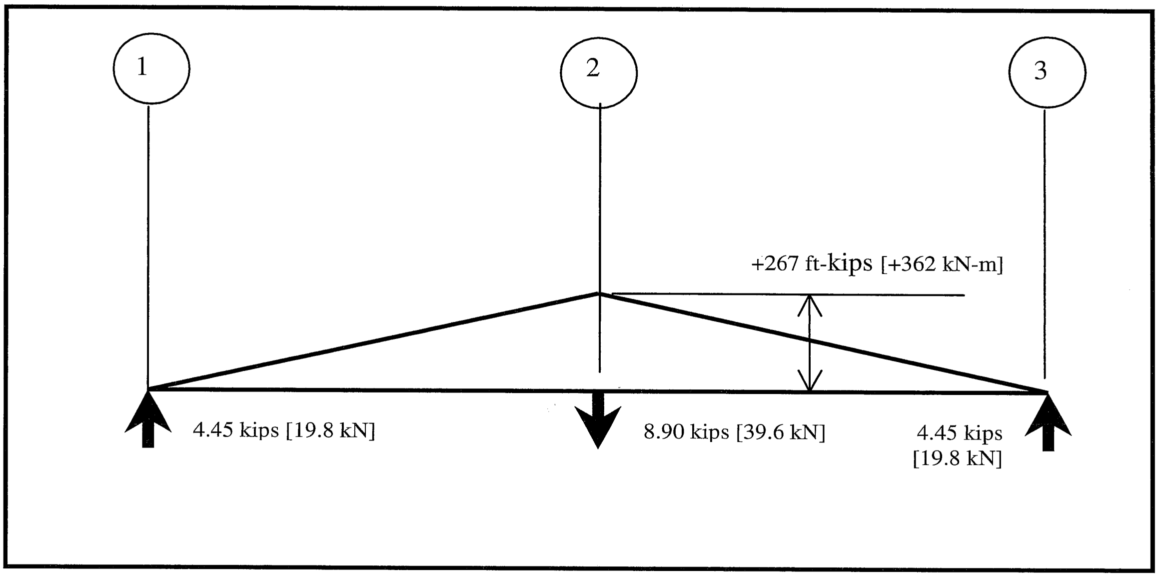


Fig. 5 - Secondary moments and reactions

compression at the bottom of the beam and tension at the top) the beam would bend in a convex upward shape, causing the beam to deflect upward (camber) at grid 2. For the deflection to be zero at grid 2, a requirement in the actual configuration, an external "secondary" reaction is required at the interior support. The direction and magnitude of this secondary reaction, and the moments it produces in the beam, are calculated as follows and are shown in Fig. 5.

The secondary moment can be determined, at any cross-section, by the following equation:

$$M_2 = M_{bal} - Fe \qquad (12)$$

where:

M_{bal} = moment at any section produced by the internal tendon loads (the "balanced" loads) acting on the concrete.

F = prestress force at same section.

e = distance between tendon *cgs* and concrete *cgc* at same section (eccentricity).

For the beam in Fig. 4 the tendon balanced load between supports is:

$$w_{bal} = \frac{8Fa}{L^2} = \frac{8x200x24}{12x60^2} = 0.889\,\text{kips/ft}\,[13.125\text{kN/m}] \qquad (13)$$

At the center support the balanced load moment is:

$$M_{bal} = \frac{0.889x60^2}{8} = 400\text{ ft - kips}\,[542\text{ kN - m}] \qquad (14)$$

The eccentricity "*e*" at the center support is 12-4=8 inches, therefore the secondary moment is:

$$M_2 = M_{bal} - Fe = 400 - 200\frac{8}{12} = +267\text{ ft - kips}\,[362\text{ kN-m}] \qquad (15)$$

The secondary reaction at supports 1 and 3 is 267/60 = 4.45 kips [19.8 kN], and the secondary reaction at the center support 2 is 8.90 kips [39.6 kN]. Since secondary moments are produced only by external reactions, secondary moment diagrams consist of straight line segments between supports. For the example beam, the secondary moment diagram is shown in Fig. 5. Moments produced by external loading are super-imposed upon this secondary moment diagram, which is present throughout the entire range of external loading from zero through failure at the limit load.

Assume that a uniform load per foot of beam is applied externally across both spans. As the load increases, at some level of applied load the "demand" moment at the center support will be equal to the capacity at that point (646 ft-kips) [876

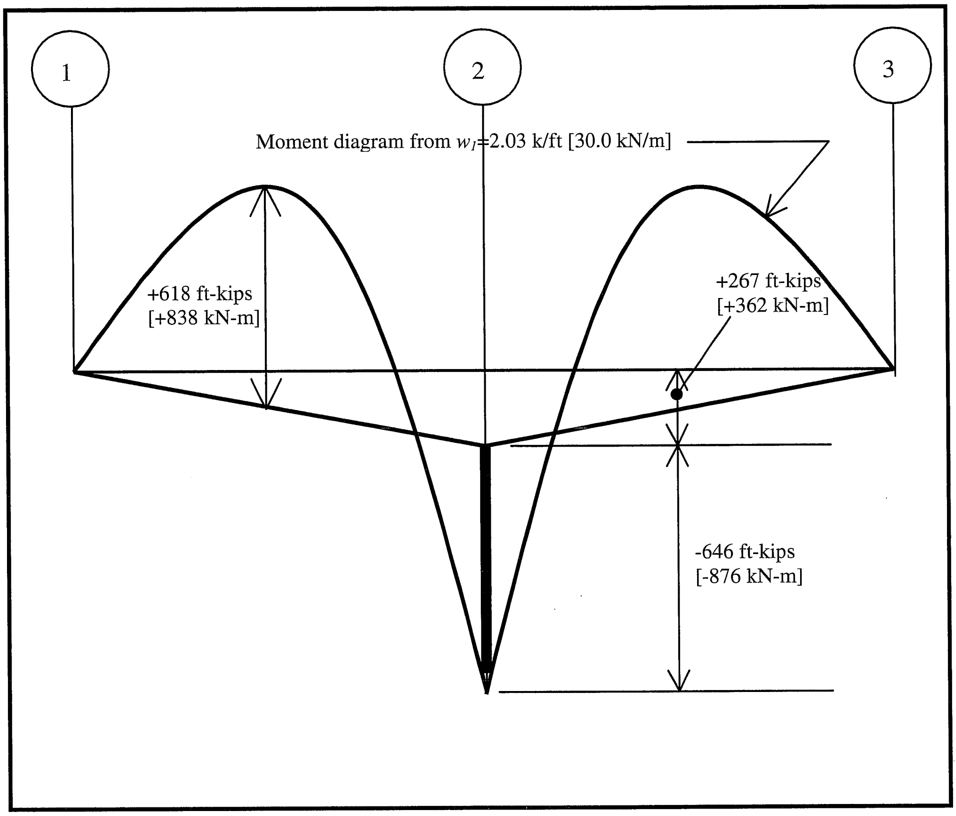


Fig. 6 - First yielding at center support at w_1=2.03k/ft [30.0 kN/m]

kN-m], and a plastic hinge will form. At that point the beam moment diagram will be as shown in Fig. 6, where the secondary moment diagram has been superimposed on the external load moment diagram.

The applied external load producing first yielding at the center support, and producing the moment diagram shown in Fig. 6, is:

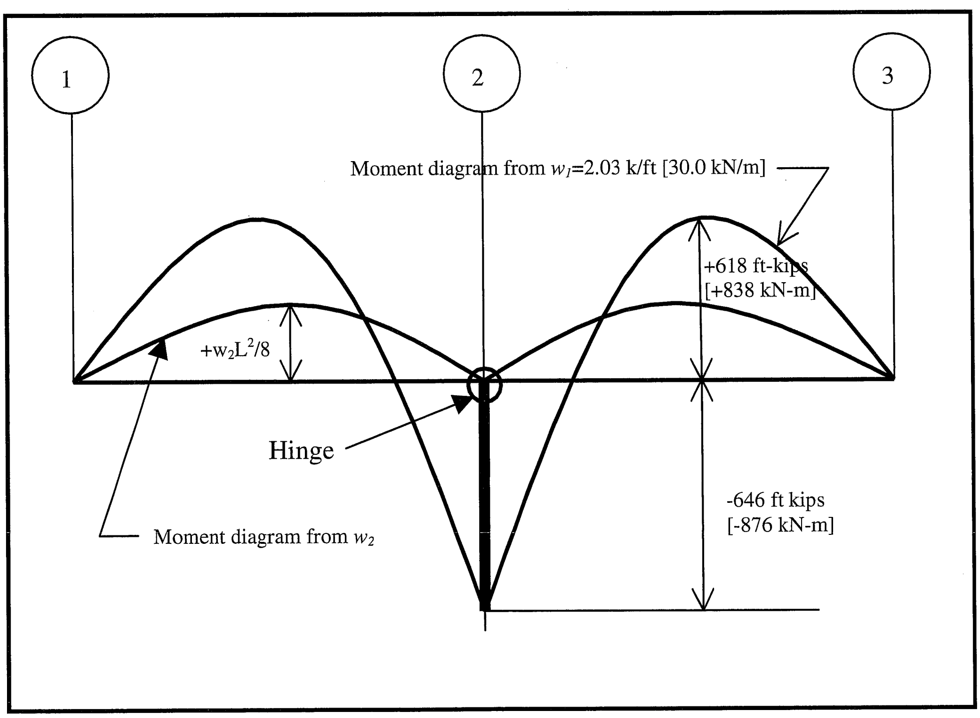


Fig. 7 - Loading beyond w_1

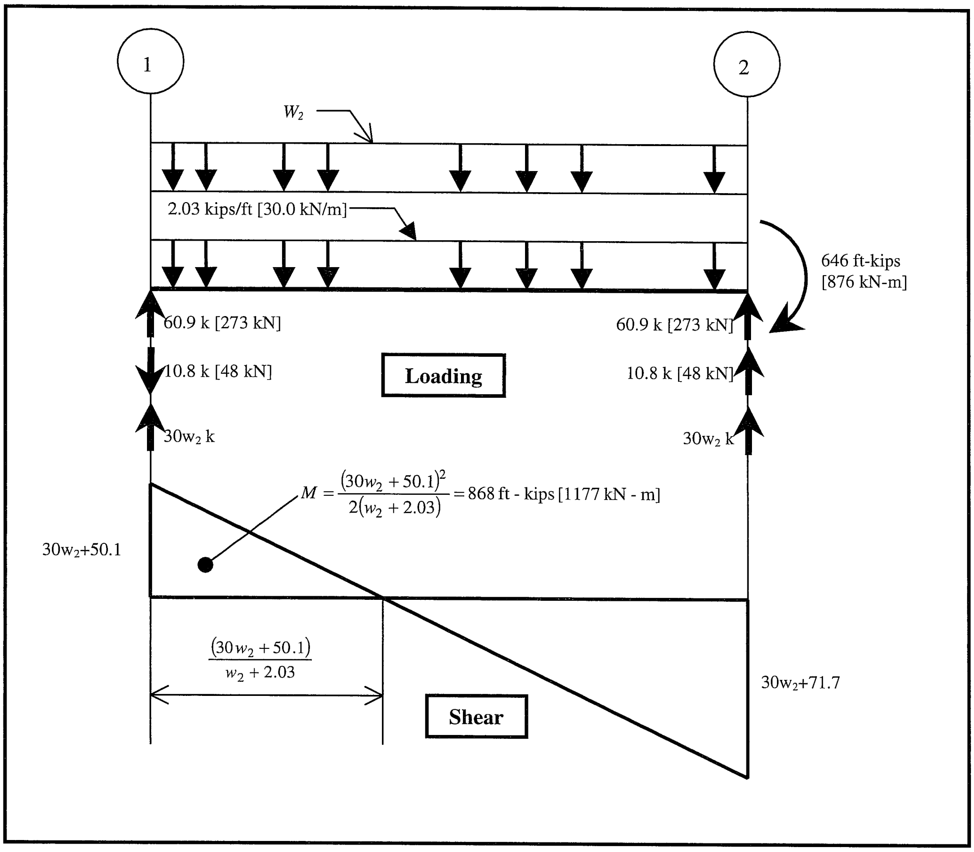


Fig. 8 - Limit load $w_2 + w_1$

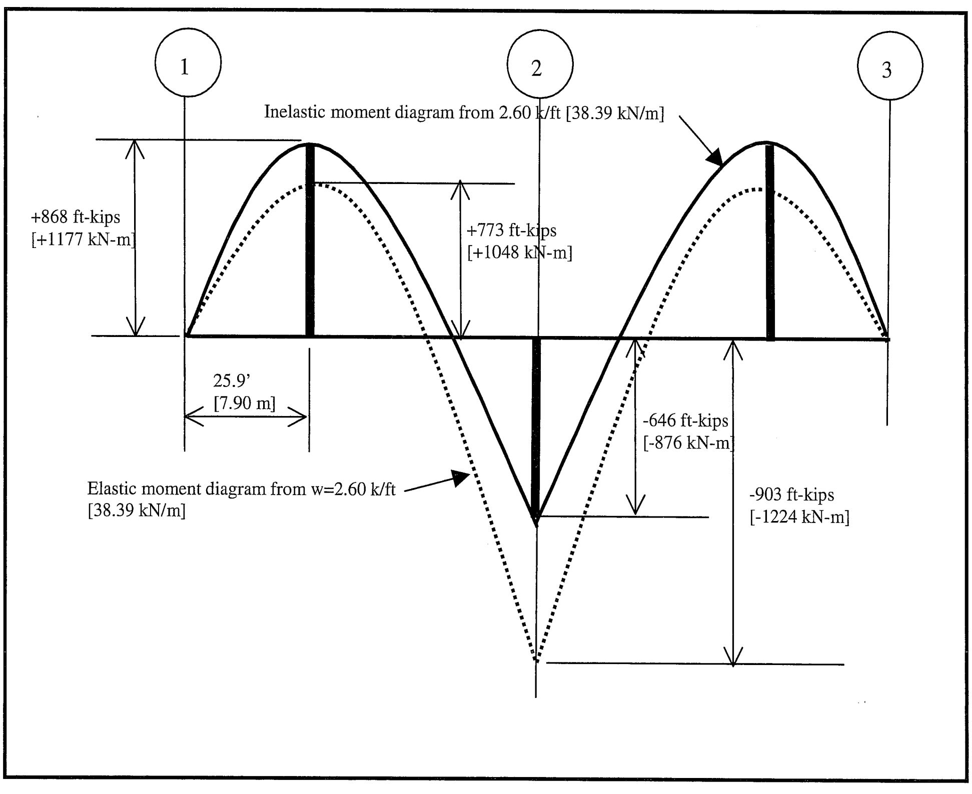


Fig. 9 - Inelastic and elastic moment diagrams with limit load

$$w_1 = \frac{8(646+267)}{60^2}$$
$$= 2.03 \text{ kips/ft}\left[30.0 \text{ kN/m}\right] \quad \text{.......................(16)}$$

Note that the elastic positive moment at midspans (618 ft-kips) [838 kN-m] under w_1 is less than the positive moment capacity (868 ft-kips) [1177 kN-m] therefore yielding at midspan has not occurred. The beam at this point is stable and can resist additional applied load, however incremental load w_2 above w_1 produces inelastic rotation but no additional moment at support 2 (a hinge has developed there), and the incremental load w_2 is resisted as two simple-span beams as shown in Fig. 7 (the baseline of the moment diagram has been rotated to horizontal for convenience).

At a certain value of w_2 the midspan applied moment will reach the positive moment capacity of 868 ft-kips [1177 kN-m], at which point the beam becomes an unstable mechanism (it can no longer respond in flexure) and the limit load has been reached. The maximum value of w_2 (0.57 kips/ft) [8.4 kN/m] can be calculated considering the statics of span 1-2 as shown in Fig. 8. The limit load for this beam is $w_1+w_2 = 2.03 + 0.57 = 2.60$ kips/ft [38.39 kN/m]. The moment diagram at limit load is shown in Fig. 9, where the *elastic* moment diagram that would have been produced by a load of 2.60 kips/ft [38.39 kN/m] is superimposed.

Important points to note from this example are:

- As the applied load increased, yielding occurred *first* at the center support, at an applied load of 2.03 k/ft [30.0 kN/m] and a negative moment of 646 ft-kips [876 kN-m]. Loading in excess of 2.03 k/ft [30.0 kN/m] produces inelastic rotation *at the center support*, with no change in moment.

- Positive moment yielding occurs at an applied "limit" load of 2.60 k/ft [38.39 kN/m]. At this point the beam becomes a mechanism and cannot carry additional load in flexure. All of the inelastic behavior occurs at the support in the negative moment region; no inelastic behavior occurs in positive moment regions.

- The limit load of 2.60 k/ft [38.39 kN/m] would produce an elastic moment of -903 ft-kips [-1224 kN-m] at the center support, including a positive secondary moment of +267 ft-kips [+362 kN-m]. The amount of inelastic redistribution required for this beam to develop full limit behavior, expressed as a percentage of the elastic moment, is:

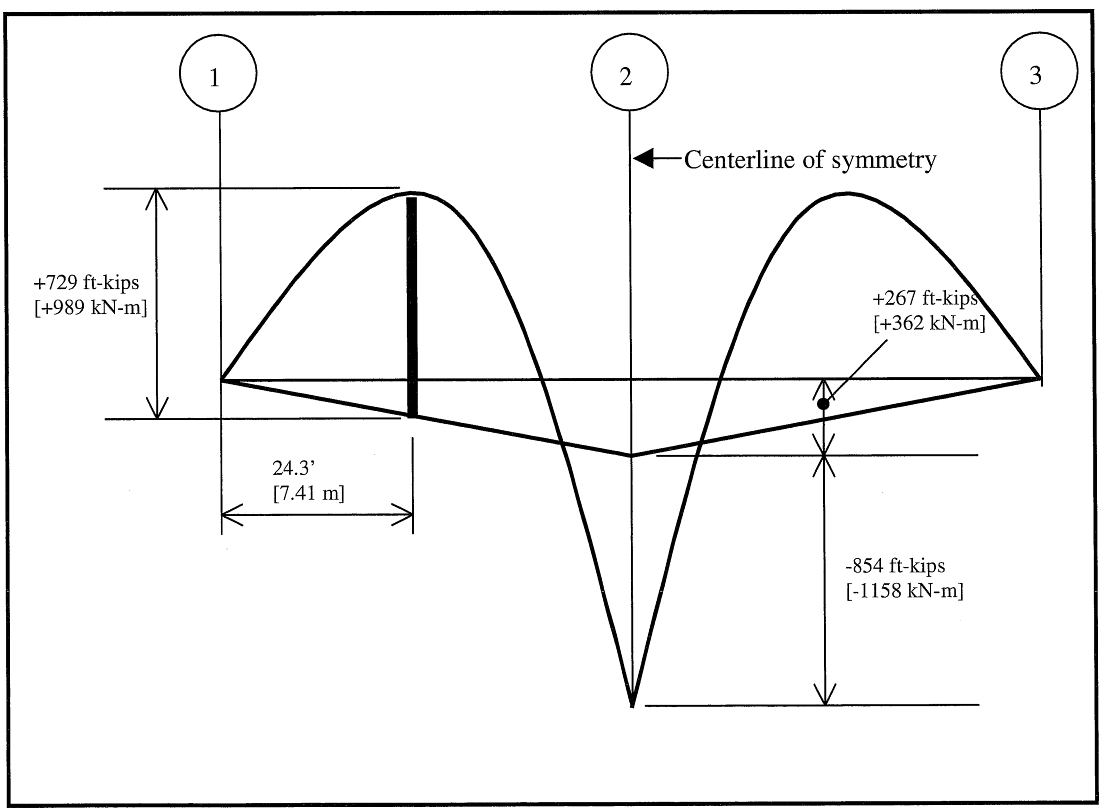


Fig. 10 - First yielding at midspan at w_1=2.49 k/ft [36.76 k/m]

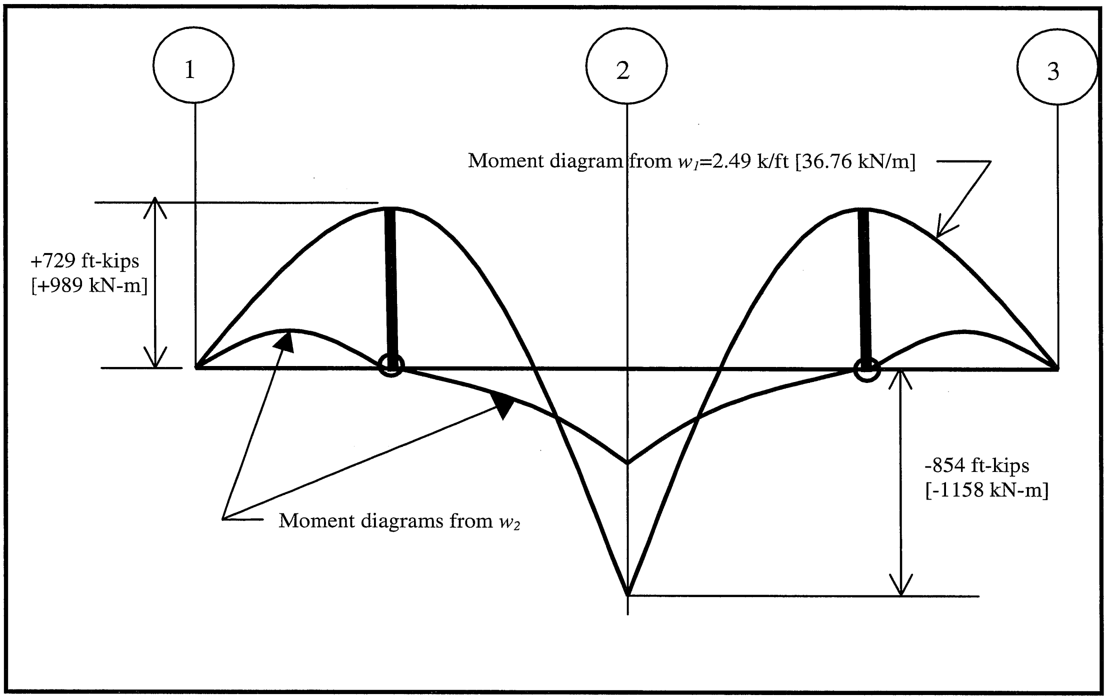
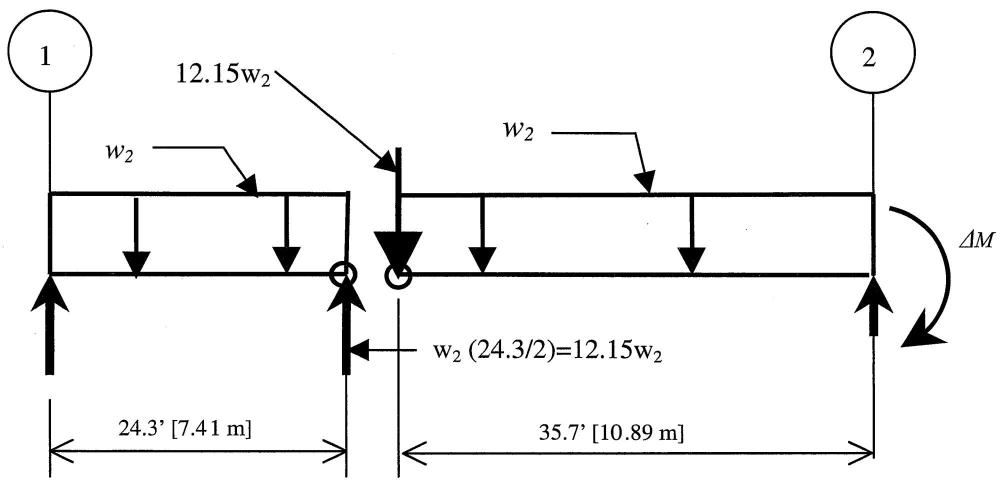


Fig.11- Loading beyond w_1



$$\Delta M = 1270 - 854 = 416 \text{ ft-kips } [564 \text{ kN-m}] = 12.15 w_2 (35.7) + \frac{w_2 (35.7)^2}{2}$$

$$1071 \; w_2 = 416$$

$$w_2 = 0.39 \text{ k/ft } [5.76 \text{ kN/m}]$$

Fig.12 - Limit load w_2

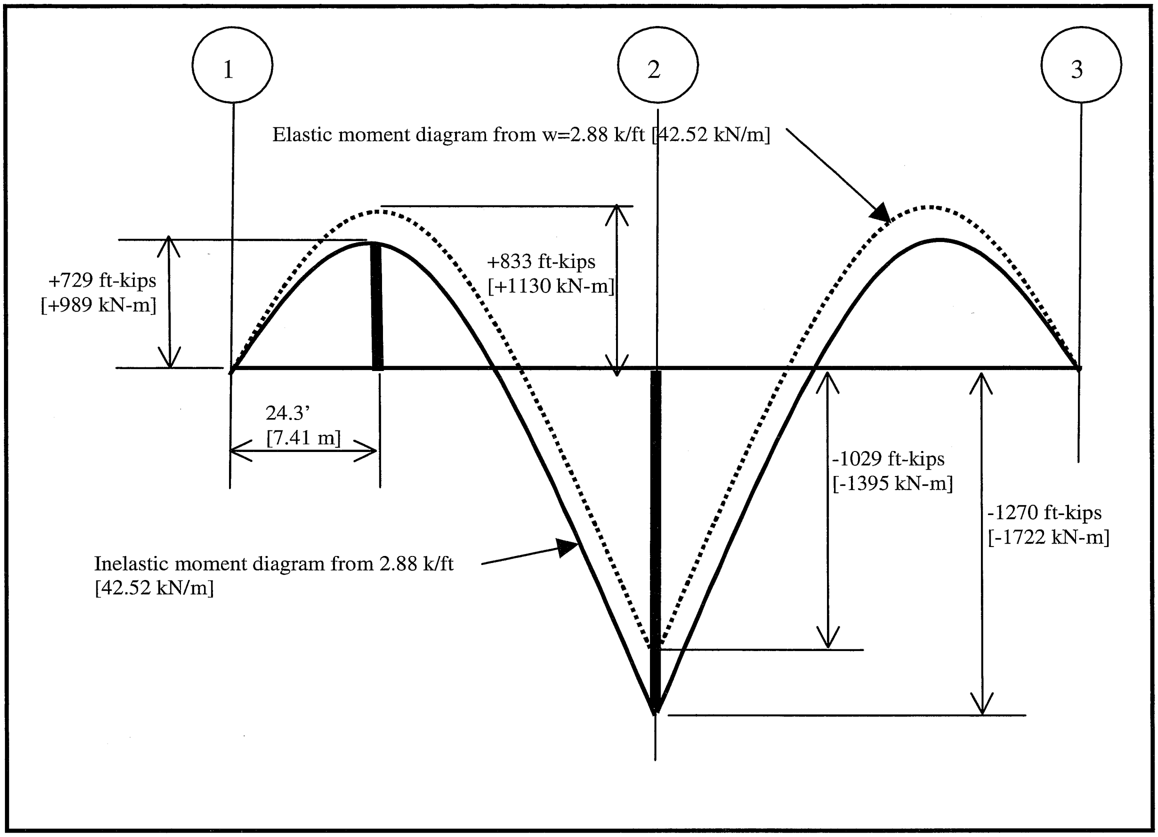


Fig. 13 - Inelastic and elastic moment diagrams with limit load

$$\%R = 100\left(1-\frac{\phi M_n}{M_e}\right) = 100\left(1-\frac{646}{903}\right) = 28.5\% \quad \text{.............................(17)}$$

- ACI 318-02 limits %R to 20%, therefore the amount of redistribution required in this beam to develop full limit load behavior would not be permitted.

- Secondary moments and reactions exist throughout the entire load range from no applied load through the limit load condition. They do not "disappear" at any load level (note that the positive secondary moments have been included algebraically in the moment diagrams of Figs. 7, 9, and 11). They have a significant effect on the amount of applied load required to produce first yielding, and the amount of inelastic rotation required at any "hinged" section between first yielding and failure.

5.2 ANALYSIS CASE 2:
Bars "A"=4-#11, Bars "B"=2-#5

This example demonstrates the behavior of a beam where yielding develops first in the positive moment region, rather than in the negative moment region at the support. For this reinforcing, the flexural capacity $ØM_n$ at support 2 is 1270 ft-kips [1722 kN-m] and at each midspan is 729 ft-kips [989 kN-m]. In calculating these capacities the tendon stress at nominal strength is 203 ksi [1400 MPa] at support 2 and 233 ksi [1607 MPa] at midspans (ACI 318-02 Eq. 18-4), and $Ø$ is 0.9 (both sections are tension-controlled with $\varepsilon_t > 0.005$, see 10.3).

Assume that a uniform load per foot of beam is applied externally across both spans. As the load increases, at some level of applied load the "demand" moment in the midspan region will be equal to the capacity at that point (729 ft-kips) [989 kN-m], and a plastic hinge will form. At that point the beam moment diagram will be as shown in Fig. 10, where the secondary moment diagram has been superimposed algebraically on the external load moment diagram.

The applied external load producing first yielding at midspan, and producing the moment diagram shown in Fig. 10, is (note that the secondary moment at the point of the hinge is +100 ft-kips) [+136 kN-m]:

$$w_1 = \frac{128(729-100)}{9x60^2} = 2.49 \text{ kips/ft } \left[36.76 \text{ kN/m}\right] \quad \text{...................(18)}$$

The elastic negative moment at the center support (854 ft-kips) [1158 kN-m] under w_1 is less than the negative moment capacity (1270 ft-kips) [1722 kN-m] therefore yielding at the center support has not occurred. The beam at this point is stable and can resist additional applied load, however incremental load w_2 above w_1 produces inelastic rotation but no additional moment at the midspan hinge, and the incremental load w_2 is resisted as a simple-span beam near the exterior supports and a cantilever off the center support as shown in Fig. 11 (the baseline of the moment diagram has been rotated to horizontal for convenience).

At a certain value of w_2 the applied support moment will reach the negative moment capacity of 1270 ft-kips [1722 kN-m], at which point the beam becomes an unstable mechanism (it can no longer respond in flexure) and the limit load has been reached. The maximum value of w_2 (0.39 kips/ft) [5.76 kN/m] can be calculated considering the statics of span 1-2 as shown in Fig. 12. The limit load for this beam is $w_1+w_2=2.49+0.39=2.88$ kips/ft [42.52 kN/m]. The moment diagram at limit load is shown in Fig. 13, where the *elastic* moment diagram that would have been produced by a load of 2.88 kips/ft [42.52 kN/m] is superimposed.

Important points to note from this example are:

- As the applied load increased, yielding occurred *first* at midspan in the positive moment region, at an applied load of 2.49 k/ft [36.76 kN/m] and a positive moment of 729 ft-kips [989 kN-m]. At this point a plastic hinge develops at a point in the span located 24.3 feet [7.41 m] from each exterior support. Loading in excess of 2.49 k/ft [36.76 kN/m] produces inelastic rotation *at the hinge*, with no change in moment.

- Negative moment yielding occurs at an applied "limit" load of 2.88 k/ft [42.52 kN/m]. At this point the beam becomes a mechanism and cannot carry additional load in flexure. All of the inelastic behavior occurs at the hinge in the positive moment region; no inelastic behavior occurs in negative moment region at the support.

- The limit load of 2.88 k/ft [42.52 kN/m] would produce an elastic moment of +833 ft-kips [+1130 kN-m] at the hinge, including a positive secondary moment of +108 ft-kips [+146 kN-m]. The amount of inelastic redistribution required in the positive moment region for this beam to develop full limit be-

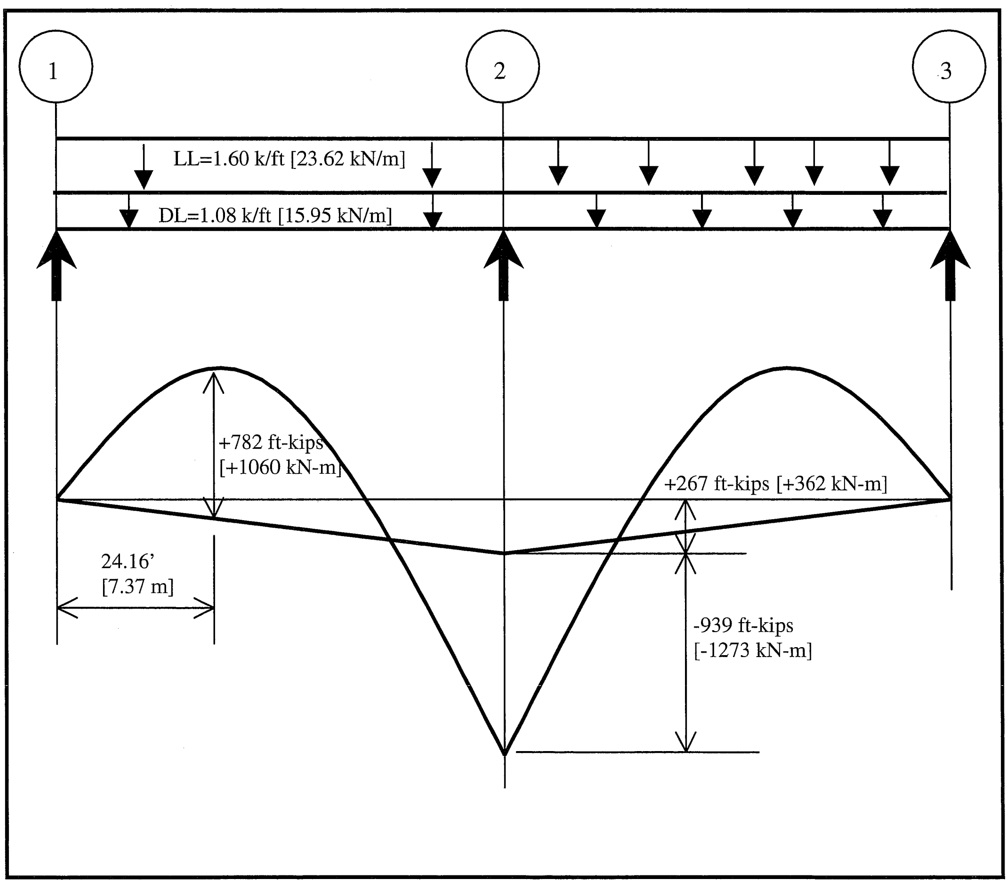


Fig. 14 - Maximum elastic negative moment as support

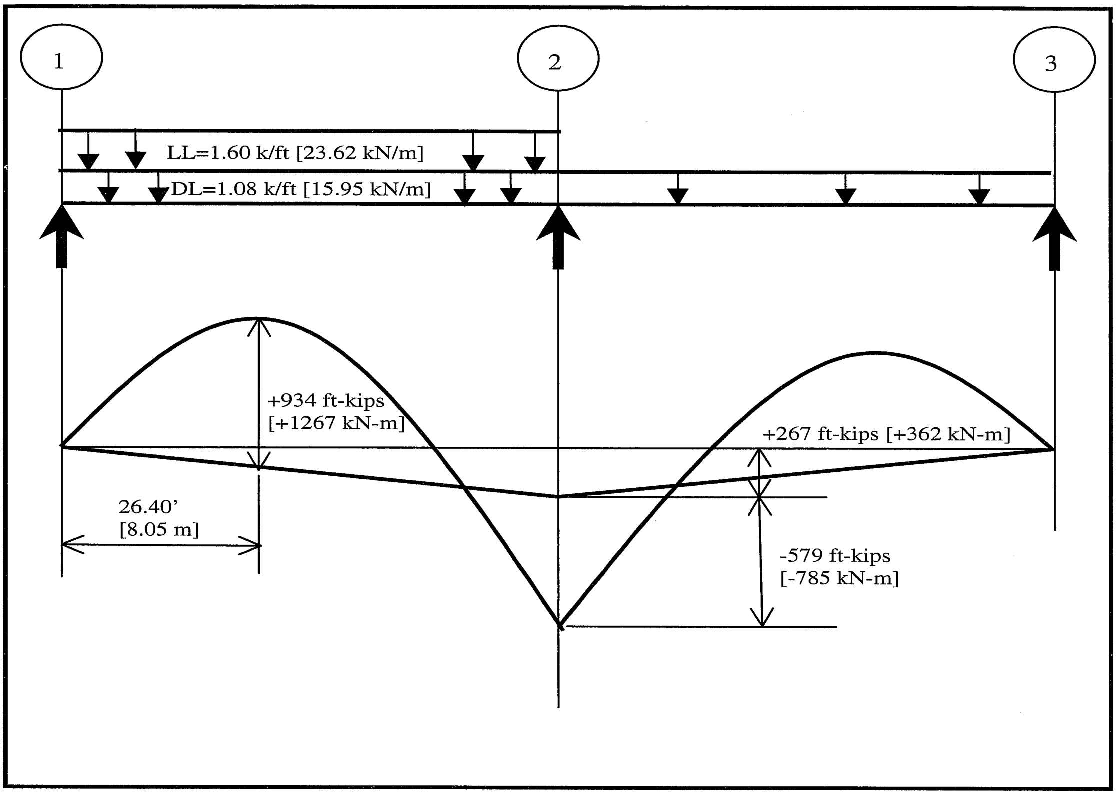


Fig. 15 - Maximum elastic positive moment in span 1-2

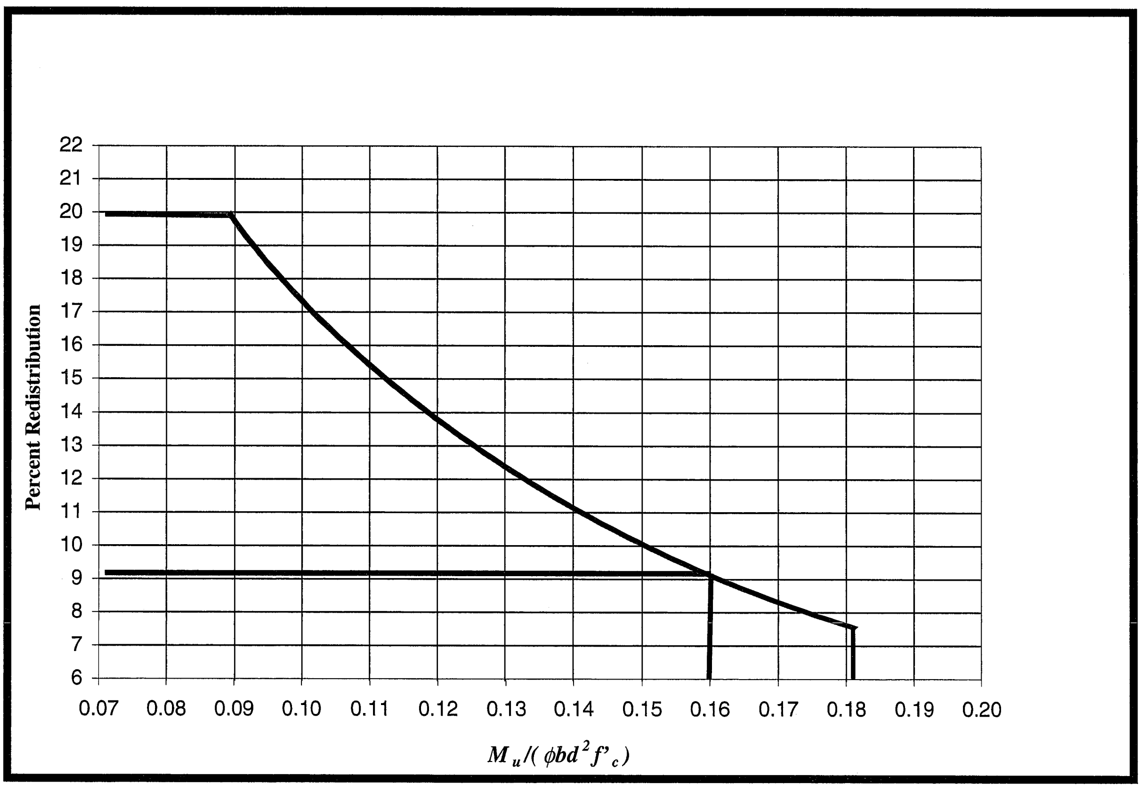


Fig. 16 - Permissible redistribution ($f'_c \leq$ 4,000 psi)

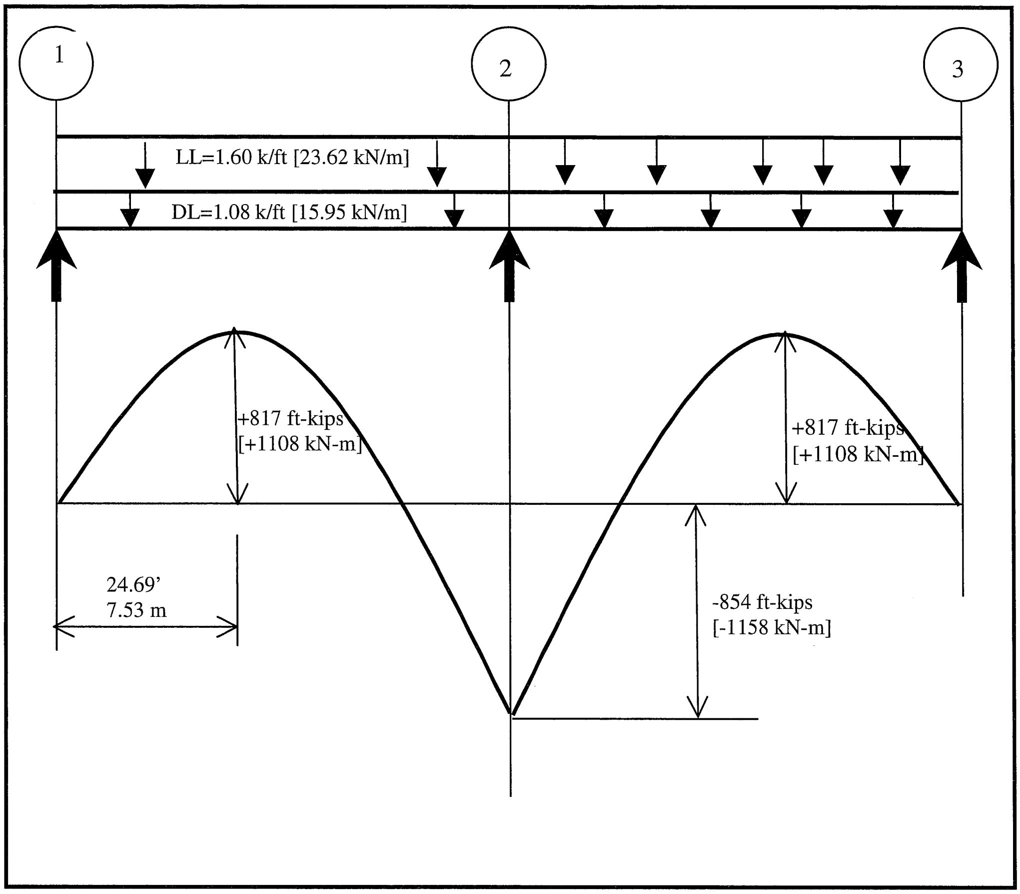


Fig. 17 - Redistributed support moment - maximum negative

havior, expressed as a percentage of the elastic moment, is:

$$\%R = 100\left(1 - \frac{\phi M_n}{M_e}\right)$$

$$= 100\left(1 - \frac{729}{833}\right) = 12.4\% \quad \text{..................... (19)}$$

- ACI 318-02 permits an increase or decrease in elastic negative moments up to 20%. In this beam the elastic negative moment is increased by 100(1270/1020-1) =12.5% in order to achieve the decrease in elastic positive moment. Note that no inelastic behavior actually occurs at the support.

- Secondary moments and reactions exist throughout the entire load range from no applied load through the limit load condition. They do not "disappear" at any load level. They have a significant effect on the amount of applied load required to produce first yielding, and the amount of inelastic rotation required at any "hinged" section between first yielding and failure.

6.0 DESIGN BY ACI 318-02

The following example demonstrates the use of moment redistribution in the *design* of a continuous post-tensioned concrete beam, specifically, the determination of design moments under factored loads at all critical sections. Design moments for the beam in Fig. 4 will be calculated assuming the prestress force and profile shown, an unfactored dead load of 0.9 kips/ft [13.29 kN/m] (including the weight of the beam), and an unfactored live load of 1.0 kip/ft [14.76 kN/m].

Calculate the factored loads:

DL = 1.2x0.9 = 1.08 kips/ft [15.95 kN/m]

LL = 1.6x1.0 = 1.60 kips/ft [23.62 kN/m]

The moment diagram for live load arranged to produce maximum negative moment at support 2 is shown in Fig. 14 (the factored load moment diagram is superimposed algebraically onto the secondary moment diagram of Fig. 5).

The moment diagram for live load arranged to produce maximum positive moment in Span 1-2 is shown in Fig. 15 (the factored load moment diagram is superimposed algebraically onto the secondary moment diagram of Fig. 5).

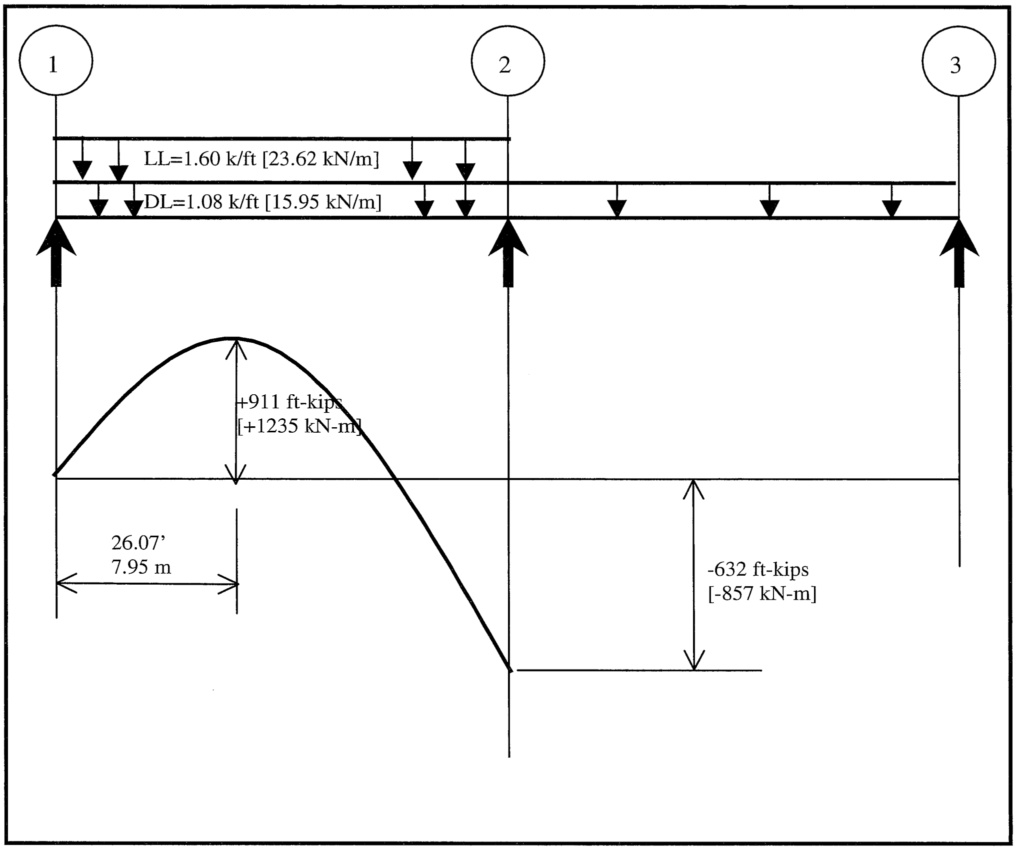


Fig.18 - Redistributed support moments - maximum positive

Based upon the elastic factored moments, the support section would require a design capacity of -939 ft-kips [-1273 kN-m], and the midspan sections would require a design capacity of +934 ft-kips [+1267 kN-m]. However ACI 318-02 permits a redistribution of elastic moments based upon a percentage increase or decrease of elastic negative support moments of $\%R=1000\varepsilon_t$, where ε_t is the net tensile strain in the tension reinforcement farthest from the compression face of the beam (see 18.10.4 and 8.4). %R is limited to 20% and ε_t must be equal to or greater than 0.075 before any redistribution is permitted. At this point in the design process, the amount of unstressed reinforcement is not known, therefore the net tensile strain is also not known. However, a mathematical relationship exists between and the quantity $M_u/(\O\, bd^2 f'_c)$, assuming a single layer of tensile reinforcement and β_1=0.85 (f'_c=4,000 psi [27.6 MPa]). That relationship is shown graphically in Fig. 16, where ε_t has been multiplied by 1000 in order to read *%R* directly on the vertical axis. For the example beam:

$$\frac{M_u}{\phi b d^2 f_c'} = \frac{939x12}{0.9x18x33^2x4} = 0.16 \quad \text{....... (20)}$$

From Fig. 16 the permissible redistribution is slightly greater than 9%. The maximum negative elastic moment from Fig. 14 can therefore be reduced to - 939 (1-9/100) =- 854 ft-kips [-1158 kN-m]. The adjusted inelastic moment diagram, consistent with a negative support moment of 854 ft-kips [1158 kN-m], is shown in Fig. 17 (the baseline of the moment diagram has been rotated to horizontal for convenience).

In Fig. 18 the support moment from the load condition producing maximum positive elastic moment in Span 1-2 has been *increased* by the same percentage, in order to reduce the maximum positive moment. The adjusted support moment for this condition is 1.09 x 579 = 632 ft-kips [857 kN-m], resulting in a *decrease* in positive moment from +934 [+1267 kN-m] to +911 ft-kips [+1235 kN-m]. Moments in the symmetrical span 2-3 are not shown in Fig. 18 since they are non-controlling.

The resulting demand moments for the beam, using moment redistribution, are -854 ft-kips [-1158 kN-m] at the support and +911 ft-kips [+1235 kN-m] at midspans, *both* reduced from the extremes of elastic moments. Fig. 19 shows, for one symmetrical span, the elastic moment diagrams for the load condition producing maximum negative moment at the center support (Curve 1); the load condition producing maximum positive moment at midspan (Curve 2); the redistributed moment diagram resulting from a 9% reduction in the maximum negative support moment (Curve 3); and the moment diagram resulting from a 9% increase in the negative support moment

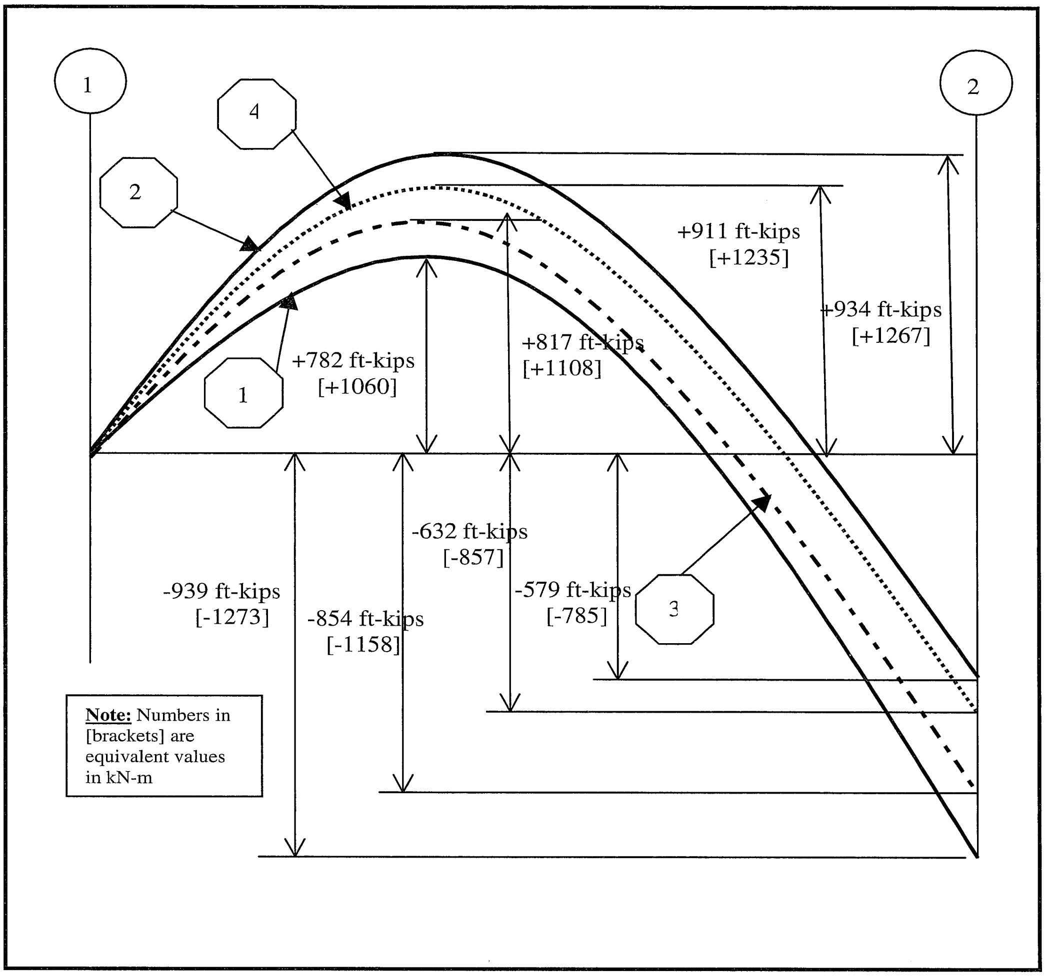


Fig. 19 - Elastic and redistributed moments

in the maximum positive moment diagram (Curve 4). It can be seen that moment redistribution has reduced the envelope of negative and positive demand moments throughout the span.

It is important when reducing both maximum positive and maximum negative moment diagrams that the redistributed moment diagrams do not "cross". In other words, when the moment diagram for maximum negative support moment (Curve 1 in Fig. 19) is transformed upward by *reducing* the elastic negative moment (Curve 3), it cannot at any point cross the moment diagram for maximum positive moment (Curve 2) transformed downwards when elastic negative moments (Curve 4) are *increased*. The two moment diagrams can converge into a single curve but they cannot exchange relative vertical positions at any point. The final envelope of redistributed positive design moments must at all points be coincident with or more positive than the final envelope of redistributed negative design moments, and of course vice versa.

7.0 OTHER MOMENT REDISTRIBUTION CONSIDERATIONS

Moment redistribution by ACI 318-02 is based upon increasing or decreasing the elastic negative moments *at supports* by 1000 ε_t%, with a maximum increase or decrease of 20% (8.4.1).

When the elastic negative support moment is *decreased*, inelastic rotation occurs in the negative moment region at the support. However when the negative support moment is *increased* above elastic levels, as permitted by 8.4.1, positive demand moments are reduced below elastic levels, and the inelastic rotation actually occurs in the positive moment regions at midspan rather than in the negative moment region at the support. The code limits the ratio of $ØM_n/M_e$ to a minimum of 0.8 at *supports* only, however it provides no direct control on the amount of inelastic rotation that can occur in the positive moment regions.

The code does state that ε_t must be equal to or greater than 0.075 *at the section at which moment is reduced* (8.4.3), but it does not limit the ratio of $ØM_n/M_e$ for positive moments if the moment reduction occurs at midspan. A given percent change in negative moment does not necessarily result in an identical percent change in positive moments. In certain cases, this can result in inelastic rotations at positive moment regions that are considerably larger than those limited by code at supports.

For example, in Fig. 3 the elastic negative moment at the supports has been increased from -2.67 units to -3.0 units, an increase of 12.4%. However the elastic positive moment has decreased from +1.33 to +1.0, a decrease of 24.8%. In this case the percent change in positive moment, representing inelastic rotation, is twice the percent change at the supports where no inelastic rotation is required.

In typical designs where live loads are required to be "skipped" to produce maximum negative and positive moments, this condition has minimal practical significance. Reasons for this are as follows:

- The moment diagram where a reduction in elastic positive moment is desirable, and positive moment hinging would therefore be required, is of course the moment diagram producing maximum elastic positive moments.
- In the moment diagram for maximum elastic positive moment, the negative and positive moments tend to be similar; in fact in many practical cases in this moment diagram the positive moment is greater than the absolute value of the negative moment, particularly in prestressed members where secondary moments tend to be positive. Thus the percentage change in positive moment will not be significantly different than the percentage change in negative moment, and in many cases will be less.
- The increased positive moment resulting from a decrease in maximum elastic negative moment will often control the design (it will be larger than the reduced positive moment in the moment diagram for maximum positive moment) and will result in less inelastic rotation in positive moment areas.
- The limitation of 20% on the change in elastic support moments, with no limitation on the percentage change in positive moments, has been in the code for many years and has been used by designers in the designs of thousands of buildings with no known detrimental effects on member performance.

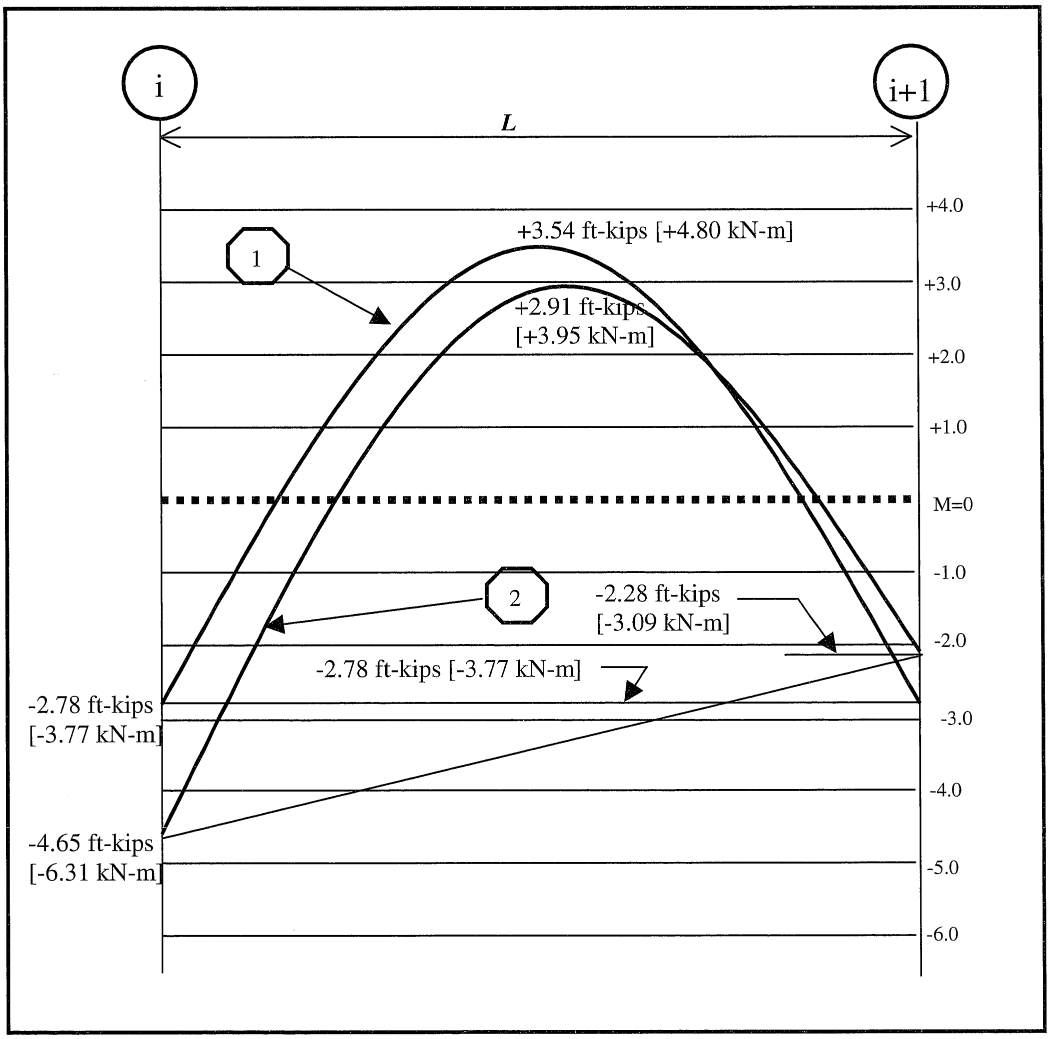


Fig. 20 - Elastic moment curves for M^+ and M^-

This can be seen in Fig. 20, which shows the elastic moment diagrams from an interior span of a typically proportioned one-way multi-span post-tensioned slab. Curve 1 is the elastic moment diagram from the loading pattern that produces maximum positive moment in the span; Curve 2 is the elastic moment diagram from the loading pattern producing maximum negative moment at the left support. Moments are expressed in units of ft-kips per foot of slab width. Both moment diagrams include a constant positive secondary moment of 0.37 ft-kips [5.46 kN-m] per foot across the span. It can be seen that the positive elastic moment (+3.54 ft-kips) [+4.80 kN-m] in the maximum positive moment diagram (Curve 1) is only about 22% larger than the positive elastic moment (+2.91 ft-kips) [+3.95 kN-m] in the maximum negative diagram (Curve 2). Thus any significant redistribution upward of the maximum negative moment diagram (normally the most cost-effective moment pattern) will leave little room for a downward redistribution of the maximum positive moment diagram (Curve 1) and very little inelastic behavior will be required in the field of the slab. It may also be seen that the magnitude of the positive moment in Curve 1 (+3.54 ft-kips) [+4.80 kN-m] is greater than the absolute magnitude of the negative moment in the same diagram (-2.78 ft-kips) [-3.77 kN-m], thus any percent change in negative moment will result in a smaller percent change in positive moment.

There is, however, one type of commonly used framing system where the same argument cannot be made, and that is the two-way slab system. In two-way slab systems where the live load is less than or equal to 75% of the dead load (not an uncommon condition), live load is not required to be skipped (13.7.6.2). In multi-span two-way slabs with spans of similar length, the single resulting moment diagram for live load applied uniformly on all spans typically contains negative moments at interior supports (in the order of $wL^2/12$) that are substantially larger than positive moments at midspans (in the order of $wL^2/24$). If elastic negative moments in an interior span of such a two-way slab were to be *increased* per code by any given percentage less than or equal to 20%, the positive elastic moment would be reduced by approximately twice that percentage, resulting in substantially more inelastic rotation at midspan than at the support when the elastic negative moment is *reduced* by the same percentage.

A further concern is the fact that minimum requirements for bonded reinforcement for moment redistribution in prestressed concrete members apply only to negative moment regions at supports, and not to midspan positive moment regions (18.10.4.1). Under certain conditions no bonded reinforcement is required in midspan positive moment areas in two-way post-tensioned slabs (18.9.3.1). It is conceivable that under 318-02 a two-way post-tensioned slab system with unbonded tendons could be designed with a midspan flexural capacity that is 40% less than the elastic positive moment demand ($ØM_n/M_e = 0.6$), and with *no* bonded reinforcement in the positive moment region.

The author is convinced that two-way post-tensioned slabs with unbonded tendons can develop the inelastic rotations required by the current code without bonded reinforcement in positive moment areas, even at the highest practical levels of inelastic demand currently permitted by the code. The following points support this opinion:

- Two-way post-tensioned slabs are generally lightly reinforced and are therefore highly ductile, with c/d_t ratios far less than the maximum tension-controlled limit of 0.375 (10.3.4 and R9.3.2.2).

- Tests have shown that two-way post-tensioned slabs with unbonded tendons develop full negative and positive "yield lines" without positive moment reinforcement[1].

- Because of the shallow moment gradient in positive moment areas, bonded reinforcement is not required to fully develop a plastic hinge.

It is the author's opinion, however, that ACI 318-02 should be modified to require a rational limitation on inelastic behavior in positive moment regions. Because of the reasons stated above, this limit can be more liberal than the limitation on inelastic behavior in negative moment regions. The following addition of Section 8.4.4 will accomplish this, and will call attention to the entire subject of positive moment redistribution:

8.4.4 – When negative moments are increased in accordance with 8.4.1, the resultant decrease in positive moments at any section shall not exceed 30%.

8.0 DISAPPEARING SECONDARY MOMENTS

ACI 318-02 addresses secondary moments in 18.10.3 and R18.10.3. While the commentary discussion of secondary moments in R18.10.3 is otherwise comprehensive and accurate, the second sentence is not correct:

"When hinges and full redistribution of moments occur to create a statically determinate structure, secondary moments disappear."

This is an unfortunate holdover from a long series of misunderstandings about secondary moments that have appeared in the ACI code and commentary since prestressed concrete first appeared in the 1963 edition. We have come a long way since the 1971 edition, where the user was told, in violation of statics, to *ignore* secondary moments in the calculation of design moments (ACI 318-71 Section 18.12).

As is shown in this paper, secondary moments do not "disappear" at any load level. Nor are they modified by the formation of any number of plastic hinges. They are present throughout the entire load range from zero to the formation of a mechanism. When the indeterminate member becomes determinate due to the formation of plastic hinges, the moment at the hinge does not become zero. The moment acting at the location of the hinge continues to act (unchanged) on the hinge after it has formed and as it rotates inelastically. Secondary moments are a part of the moment acting on the hinge, along with moments from applied external loading, and they continue to act on the hinge throughout the entire post-yielding load range. Secondary moments affect the demand moments and the required amount of inelastic rotations required at any load level, elastic or inelastic.

The author recommends that the cited second sentence in R18.10.3 be removed from the ACI code to avoid any misunderstandings about the statics and effects of secondary moments.

9.0 SUMMARY AND CLOSURE

This paper has described the mechanics of moment redistribution in indeterminate continuous concrete members. Examples of moment redistribution were presented for both the analysis and the design of a continuous concrete beam. The beam was prestressed in order to demonstrate the effects and interactions of secondary moments and moment redistribution. Recommendations are made for changes to ACI 318 that will address certain aspects of moment redistribution and secondary moments not adequately addressed in the current edition.

REFERENCES

1 *"Design of Post-Tensioned Slabs"*, Post-Tensioning Institute, Phoenix, Arizona, 1984, 54 pp.